KINEMATICS

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Contents

Summary

This chapter provides the basic infrastructure necessary for a rigorous study of continuum mechanics. The topics include tensor algebra and analysis, geometry and motion of continuous bodies, and singular surfaces. The concepts of tensor algebra and analysis form the language of continuum mechanics and it therefore becomes essential to have a good familiarity with them. A continuous body can demonstrate highly complicated deformations, thus requiring precise notions to characterize their geometry and motion. Singular surfaces are surfaces across which variables such as velocity and deformation suffer jump discontinuities. Understanding their kinematical behavior is a starting point in the study of many important phenomena including the propagation of shock waves, phase fronts, and grain boundaries.

1. Preliminaries

The following notation is adopted in which $\mathcal{V}$ is the translation (vector) space of a real three-dimensional Euclidean point space $\mathcal{E}$:

Lin: the linear space of linear transformations (tensors) from $\mathcal{V}$ to $\mathcal{V}$.

$InvLin$: the group of invertible tensors.

$Sym = \{\mathbf{A} \in Lin : \mathbf{A} = \mathbf{A}^{\mathrm{T}}\}$, where superscript T denotes the transpose: linear space of symmetric tensors; also, the linear operation of symmetrization on Lin.

$Sym^{+} = \{\mathbf{A} \in Sym : \mathbf{u} \cdot \mathbf{A}\mathbf{u} > 0\}$ for $\mathbf{u} \neq 0, \mathbf{u} \in \mathcal{V}$: the positive-definite tensors.

$Skw = \{\mathbf{A} \in Lin : \mathbf{A}^{\mathrm{T}} = -\mathbf{A}\}$: the linear space of skew tensors; also, the linear operation

of skew-symmetrization on Lin.
$Orth = \{\mathbf{A} \in InvLin : \mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}\}$, where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$: the group of orthogonal tensors.
$Orth^{+} = \{\mathbf{A} \in Orth : J_A = 1\}$: the group of rotations.

Here and in the following chapter on balance laws, both indicial notation as well as bold notation are used to represent vector and tensor fields. The components in the indicial notation are written with respect to the Cartesian coordinate system. Indices denoted with roman alphabets vary from one to three but those denoted with Greek alphabets vary from one to two. Einstein's summation convention is assumed unless an exception is explicitly stated. Let e_{ijk} be the three dimensional permutation symbol, i.e. $e_{ijk} = 1$ or $e_{ijk} = -1$ when (i, j, k) is an even or odd permutation of $(1,2,3)$, respectively, and $e_{ijk} = 0$ otherwise.

The determinant and cofactor of $\mathbf{A}$ are denoted by J_A and $\mathbf{A}^*$, respectively, where $\mathbf{A}^* = J_A\mathbf{A}^{-\mathrm{T}}$ if $\mathbf{A} \in InvLin$. It follows easily that $(\mathbf{AB})^* = \mathbf{A}^*\mathbf{B}^*$. Further, Lin is equipped with the Euclidean inner product and norm defined by $\mathbf{A}\cdot\mathbf{B} = \mathrm{tr}(\mathbf{AB}^{\mathrm{T}})$ and $|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A}$, respectively, where $\mathrm{tr}(\cdot)$ is the trace operator. We make frequent use of relations like $\mathbf{A}\cdot\mathbf{BC} = \mathbf{AC}^{\mathrm{T}}\cdot\mathbf{B} = \mathbf{C}^{\mathrm{T}}\cdot\mathbf{A}^{\mathrm{T}}\mathbf{B}$ and $\mathbf{AB}\cdot\mathbf{CD} = \mathbf{ABD}^{\mathrm{T}}\cdot\mathbf{C}$, etc., which follow easily from $\mathrm{tr}\,\mathbf{A} = \mathrm{tr}\,\mathbf{A}^{\mathrm{T}}$ and $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$. It is well known that $Lin = Sym \oplus Skw$, the direct sum of *Sym* and *Skw*. The tensor product $\mathbf{a}\otimes\mathbf{b}$ of vectors $\{\mathbf{a},\mathbf{b}\} \in \mathcal{V}$ is defined by $(\mathbf{a}\otimes\mathbf{b})\mathbf{v} = (\mathbf{b}\cdot\mathbf{v})\mathbf{a}$ for all $\mathbf{v}$ in $\mathcal{V}$, where $\mathbf{b}\cdot\mathbf{v}$ is the standard inner product of vectors.

2. Body, Configurations, and Motion

The geometrical structure of a physical body is independent of a frame of reference, and therefore the *body* (in continuum mechanics) is usually taken to be a three dimensional differential manifold. We denote such a manifold by $\mathfrak{B}$ and call its elements material points. At every material point $X \in \mathfrak{B}$ we have an associated tangent space $\mathcal{T}_X$ which is a three dimensional vector space representing a neighborhood of X. On the other hand, the body is observed and tested in a (three dimensional) Euclidean frame of reference $\mathcal{E}$, which requires us to endow the body $\mathfrak{B}$ with a class C of bijective mappings, $\chi : \mathfrak{B} \to \mathcal{E}_\chi$ (the subscript χ is used to indicate the mapping employed).
We call these mappings the *configurations* of the body $\mathfrak{B}$. The spatial position $\chi(X) \in \mathcal{E}_\chi$ denotes the *place* which a material point $X \in \mathfrak{B}$ occupies in $\mathcal{E}_\chi$. The translation space of $\mathcal{E}_\chi$ is a three dimensional inner product space, and is denoted by $\mathcal{V}_\chi$.

We introduce a fixed *reference configuration*, relative to which the notions of displacement and strain can be defined. Let $\boldsymbol{\kappa} \in C$ be a reference configuration. The configuration $\boldsymbol{\kappa}$ need not be a configuration occupied by $\mathfrak{B}$ at any time and therefore

$\boldsymbol{\kappa}$ can be arbitrary as long as it belongs to C.

The *motion* of a body $\mathfrak{B}$ is defined as a one-parameter family of configurations, $\chi_t : \mathfrak{B} \times \mathbb{R} \to \mathcal{E}_\chi$. Such a motion assigns a place $\mathbf{x} \in \mathcal{E}_\chi$ to the material point $X \in \mathfrak{B}$ at time t. We write this as

$$\mathbf{x} = \chi_t(X) \equiv \chi(X,t). \quad (1)$$

The reference configuration $\boldsymbol{\kappa}$ assigns a place $\mathbf{X} \in \mathcal{E}_\kappa$ to X, so we can express $\mathbf{x}$ as a function of $\mathbf{X}$,

$$\mathbf{x} = \chi(\boldsymbol{\kappa}^{-1}(\mathbf{X}),t) \equiv \chi_\kappa(\mathbf{X},t), \quad (2)$$

where $\chi_\kappa : \mathcal{E}_\kappa \times \mathbb{R} \to \mathcal{E}_\chi$ denotes a mapping from the reference configuration to the configuration of the body at time t.

The *displacement* $\mathbf{u} : \mathfrak{B} \times \mathbb{R} \to \mathcal{V}$ ($\mathcal{V}$ can be identified with either $\mathcal{V}_\chi$ or $\mathcal{V}_\kappa$) of a material point X with respect to the reference configuration κ at time t is defined as

$$\mathbf{u}(X,t) = \chi(X,t) - \kappa(X). \quad (3)$$

The particle *velocity* $\mathbf{v} : \mathfrak{B} \times \mathbb{R} \to \mathcal{V}_\chi$ and the particle *acceleration* $\mathbf{a} : \mathfrak{B} \times \mathbb{R} \to \mathcal{V}_\chi$ are defined as

$$\mathbf{v}(X,t) = \frac{\partial}{\partial t}\chi(X,t) \quad (4)$$

and

$$\mathbf{a}(X,t) = \frac{\partial^2}{\partial t^2}\chi(X,t), \quad (5)$$

respectively. Displacement, particle velocity and particle acceleration can all be alternatively expressed as functions on $\kappa(\mathfrak{B})$ by using the inverse $\kappa^{-1} : \mathcal{E}_\kappa \to \mathfrak{B}$. Such functions exist in a one-to-one relation with the functions expressed in the equations above. We write

$$\hat{\mathbf{u}}(\mathbf{X},t) \equiv \mathbf{u}(\kappa^{-1}(\mathbf{X}),t)$$

$$\hat{\mathbf{v}}(\mathbf{X},t) \equiv \mathbf{v}(\kappa^{-1}(\mathbf{X}),t) \quad (6)$$

$$\hat{\mathbf{a}}(\mathbf{X},t) \equiv \mathbf{a}(\kappa^{-1}(\mathbf{X}),t).$$

We can similarly write these functions as

$$\tilde{\mathbf{u}}(\mathbf{x},t) \equiv \mathbf{u}(\chi_t^{-1}(\mathbf{x}),t)$$

$$\tilde{\mathbf{v}}(\mathbf{x},t) \equiv \mathbf{v}(\chi_t^{-1}(\mathbf{x}),t) \tag{7}$$

$$\tilde{\mathbf{a}}(\mathbf{x},t) \equiv \mathbf{a}(\chi_t^{-1}(\mathbf{x}),t).$$

We define the *material time derivative* as the derivative of a function with respect to time for a fixed material point. For an arbitrary scalar function $f : \mathfrak{B} \times \mathbb{R} \to \mathbb{R}$, we denote its material time derivative as $\dot{f}$. Thus,

$$\dot{f} = \frac{\partial}{\partial t} f(X,t)\big|_X , \tag{8}$$

where the notation $\big|_X$ denotes the evaluation of the derivative at a fixed X. If f is instead given in terms of $\mathbf{x}$, i.e. if $f = \tilde{f}(\chi(X,t),t)$, we write

$$\dot{f} = \frac{\partial}{\partial t} \tilde{f}(\mathbf{x},t)\big|_{\mathbf{x}} + (\operatorname{grad} \tilde{f}) \cdot \mathbf{v}, \tag{9}$$

where $\frac{\partial}{\partial t} \tilde{f}(\mathbf{x},t)\big|_{\mathbf{x}}$ is the *spatial time derivative* (at a fixed $\mathbf{x}$) and $\operatorname{grad} \tilde{f}$ is the spatial gradient (gradient is defined below). Therefore, if the particle velocity is a function of spatial position $\mathbf{x}$, then the particle acceleration is $\tilde{\mathbf{a}} = \frac{\partial}{\partial t} \tilde{\mathbf{v}}(\mathbf{x},t)\big|_{\mathbf{x}} + \mathbf{L}\mathbf{v}$, where $\mathbf{L} = \operatorname{grad} \tilde{\mathbf{v}}$ is the spatial velocity gradient.

Derivatives

By *fields* we mean scalar, vector and tensor valued functions defined on position ($\mathbf{x}$ or $\mathbf{X}$) and time (t). In the following we are mainly concerned with the derivatives with respect to the position and therefore dependence of fields on time is suppressed.

A scalar-valued field $\phi(\mathbf{X}) : \mathcal{E}_\kappa \to \mathbb{R}$ is differentiable at $\mathbf{X}_0 \in \mathcal{U}(\mathbf{X}_0)$, where $\mathcal{U}(\mathbf{X}_0) \subset \mathcal{E}_\kappa$ is an open neighborhood of $\mathbf{X}_0$, if there exists a unique $\mathbf{c} \in \mathcal{V}_\kappa$ such that

$$\phi(\mathbf{X}) = \phi(\mathbf{X}_0) + \mathbf{c}(\mathbf{X}_0) \cdot (\mathbf{X} - \mathbf{X}_0) + o\left(|\,\mathbf{X} - \mathbf{X}_0\,|\right), \tag{10}$$

where $\frac{o(\epsilon)}{\epsilon} \to 0$ as $\epsilon \to 0$. We call $\mathbf{c}(\mathbf{X}_0) = \nabla\phi|_{\mathbf{X}_0}$ (or $\nabla\phi(\mathbf{X}_0)$) the *gradient* of ϕ at $\mathbf{X}_0$. Consider a curve $\mathbf{X}(u)$ in $\mathcal{E}_\kappa$ parameterized by $u \in \mathbb{R}$. Let $\psi(u) = \phi(\mathbf{X}(u))$ and $\mathbf{X}_1 = \mathbf{X}(u_1), \mathbf{X}_0 = \mathbf{X}(u_0)$ for $\{u_1, u_0\} \in \mathbb{R}$. Then from (10),

$$\psi(u_1) - \psi(u_0) = \nabla\phi(\mathbf{X}_0) \cdot (\mathbf{X}_1 - \mathbf{X}_0) + o\left(|\,\mathbf{X}_1 - \mathbf{X}_0\,|\right). \tag{11}$$

Moreover $\mathbf{X}_1 - \mathbf{X}_0 = \mathbf{X}'(u_0)(u_1 - u_0) + o(|u_1 - u_0|)$, where $\mathbf{X}'(u_0)$ is the derivative of $\mathbf{X}$ with respect to u at $u = u_0$. Therefore, $|\mathbf{X}_1 - \mathbf{X}_0| = O(|u_1 - u_0|)$ and consequently we can rewrite (11)

$$\frac{\psi(u_1) - \psi(u_0)}{u_1 - u_0} = \nabla\phi(\mathbf{X}_0) \cdot \mathbf{X}'(u_0) + \frac{o(|u_1 - u_0|)}{u_1 - u_0}. \tag{12}$$

For $u_1 \to u_0$ we obtain *the chain rule*, $\psi'(u_0) = \nabla\phi(\mathbf{X}(u_0)) \cdot \mathbf{X}'(u_0)$, which can also be expressed as $\frac{d\phi}{du} = \nabla\phi(\mathbf{X}) \cdot \frac{d\mathbf{X}}{du}$ or

$$d\phi(\mathbf{X}) = \nabla\phi(\mathbf{X}) \cdot d\mathbf{X}. \tag{13}$$

A vector-valued field $\mathbf{v}(\mathbf{X}) : \mathcal{E}_\kappa \to \mathcal{V}$ is differentiable at $\mathbf{X}_0 \in \mathcal{U}(\mathbf{X}_0)$ if there exists a unique tensor $\mathbf{l} : \mathcal{V}_\kappa \to \mathcal{V}$ such that

$$\mathbf{v}(\mathbf{X}) = \mathbf{v}(\mathbf{X}_0) + \mathbf{l}(\mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0) + \mathbf{r}, \tag{14}$$

where $|\mathbf{r}| = o\left(|\mathbf{X} - \mathbf{X}_0|\right)$. We call $\mathbf{l}(\mathbf{X}_0) = \nabla\mathbf{v}|_{\mathbf{X}_0}$ (or $\nabla\mathbf{v}(\mathbf{X}_0)$) the *gradient* of $\mathbf{v}$ at $\mathbf{X}_0$. The chain rule in this case can be obtained following the procedure preceding Eq. (13):

$$d\mathbf{v}(\mathbf{X}) = (\nabla\mathbf{v})d\mathbf{X}. \tag{15}$$

The *divergence* of a vector field is a scalar defined by

$$\mathrm{Div}\,\mathbf{v} = \mathrm{tr}(\nabla\mathbf{v}). \tag{16}$$

The *curl* of a vector field is a vector defined by

$$(\mathrm{Curl}\,\mathbf{v}) \cdot \mathbf{c} = \mathrm{Div}(\mathbf{v} \times \mathbf{c}) \tag{17}$$

for any fixed $\mathbf{c} \in \mathcal{V}$.

Differentiability of a tensor-valued function is defined in a similar manner. In particular, for a tensor $\mathbf{A}(\mathbf{X}) : \mathcal{E}_\kappa \to Lin$, we write

$$d\mathbf{A}(\mathbf{X}) = (\nabla\mathbf{A})d\mathbf{X}. \tag{18}$$

The *divergence* of $\mathbf{A}$ is the vector defined by

$$(\mathrm{Div}\,\mathbf{A}) \cdot \mathbf{c} = \mathrm{Div}(\mathbf{A}^{\mathrm{T}}\mathbf{c}) \tag{19}$$

for any fixed $\mathbf{c} \in \mathcal{V}$. The superscript T denotes the transpose. The *curl* of $\mathbf{A}$ is the tensor defined by

$$(\mathrm{Curl}\,\mathbf{A})\mathbf{c} = \mathrm{Curl}(\mathbf{A}^{\mathrm{T}}\mathbf{c}) \tag{20}$$

for any fixed $\mathbf{c} \in \mathcal{V}$.

Finally, if the fields are expressed as functions of $\mathbf{x}$ rather than $\mathbf{X}$, the various definitions above remain valid. We instead denote the gradient, divergence and curl operators by grad, div, and curl, respectively.

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Biographical Sketches

Anurag Gupta He received B.Tech. in Civil Engineering from Indian Institute of Technology at Roorkee in 2002, M.S. in Civil and Environmental Engineering from University of California at Berkeley in 2003, and Ph.D. in Mechanical Engineering from University of California at Berkeley in 2008. His thesis dealt with plastic deformation in solids with interfaces. Currently, he is an Assistant Professor in the department of Mechanical Engineering at Indian Institute of Technology, Kanpur, India. His research interests include plasticity theory, dynamics of defects in solids, thermodynamics of irreversible processes, and thin films.

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